# Medical Image Analysis

CS 778 / 578

Computer Science and Electrical Engineering Dept.
West Virginia University

February 14, 2011

# Outline

1. Basic active contour behavior
2. Internal forces
3. Image-based forces
4. Constraints
5. Numerical implementation
6. Weaknesses
7. Extensions to the original paper

# Outline


1. Basic active contour behavior
   - Geometry of curves
2. Internal forces
   - Membrane spline energy
   - Thin-plate spline energy
3. Image-based forces
4. Constraints
5. Numerical implementation
6. Weaknesses
7. Extensions to the original paper

# Basic active contour behavior

An **active contour** (or **snake**) is an energy minimizing parametric curve which evolves according to external constraints and is influenced by image forces which pull it toward features of interest.

- The exact energy functionals involved will depend on which features are of interest.
- We will impose membrane and thin-plate spline smoothness constraints on the snakes, and also allow for user intervention.

# Energy Functional

Behavior is governed by the energy functional

$$E^*_{snake} = \int_0^1 E_{int}(v(s)) + E_{image}(v(s)) + E_{con}(v(s))\ ds$$

- $v(s)$ is the parametric curve
- $E_{int}$ represents the smoothness constraints
- $E_{image}$ represents image data constraints
- $E_{con}$ represents user input

# Internal Energy

A combination of membrane and thin-plate spline smoothness

$$E_{int} = \frac{1}{2}(\alpha(s)||v_s(s)||^2 + \beta(s)||v_{ss}(s)||^2)$$

- This model allows the weights $\alpha, \beta$ to vary along the length of the curve.
- Setting $\beta(s) = 0$ allows a discontinuity to develop at $s$.

### Definition

**Parametric Curve** : A vector valued function from some interval of the real line to Euclidean space.

$$c(p) = \begin{bmatrix} x(p) \\ y(p) \end{bmatrix}$$

where $p \in [a, b]$.

If $c(a) = c(b)$ the curve is **closed**.
If $||c'(p)|| = 1$ the curve is **parameterized by arclength**.

# Tangent and Normal vector

## Unit Tangent Vector

$$T(p) = \frac{c'(p)}{||c'(p)||}$$

## Unit Normal Vector

$$N(p) = \frac{T'(p)}{||T'(p)||}$$

$T(p) \perp N(p) \rightarrow T(p) \cdot N(p) = 0.$

$$N(p) = \begin{bmatrix} -t_y \\ t_x \end{bmatrix} \text{or} \begin{bmatrix} t_y \\ -t_x \end{bmatrix}$$

# Outline


1. Basic active contour behavior
   - Geometry of curves
2. Internal forces
   - Membrane spline energy
   - Thin-plate spline energy
3. Image-based forces
4. Constraints
5. Numerical implementation
6. Weaknesses
7. Extensions to the original paper

In general, we may want to minimize a combination of the two energies:

$$\alpha E_{MEM}(c) + \beta E_{TPS}(c) = \int_0^l \alpha||c'(s)||^2 + \beta||c''(s)||^2\, ds$$

We will show that the minimization conditions are

$$-\frac{d}{ds}\alpha c'(s) + \frac{d^2}{ds^2}\beta c''(s) = 0$$

$$-\alpha c''(s) + \beta c''''(s) = 0$$

# Variational Calculus

The conditions for minimizing

$$\min_c \int_\Omega F(s, c(s), c'(s), c''(s))\, ds$$

are

$$F_c - \frac{d}{ds} F_{c'(s)} + \frac{d^2}{ds^2} F_{c''(s)} = 0$$

The evolution equation that will satisfy this condition when steady-state has been reached

$$\frac{\partial c}{\partial t} = -F_c + \frac{d}{ds} F_{c'(s)} - \frac{d^2}{ds^2} F_{c''(s)}$$

# Membrane spline energy of c(s)

$$E_{MEM}(c) = \int_0^L ||c'(s)||^2 \, ds = \int_0^L x'(s)^2 + y'(s)^2 \, ds$$

For an arclength parameterization, minimizing $E_{MEM}(c)$ is equivalent to minimizing the length of c.

Applying variational calculus to

$$\min_{x(s),y(s)} \int_0^L x'(s)^2 + y'(s)^2 \, ds$$

gives two Euler-Lagrange conditions

$$\begin{aligned} \frac{d}{ds}(2x'(s)) &= 0 \\ \frac{d}{ds}(2y'(s)) &= 0 \end{aligned}$$

# Membrane spline energy of c(s)

The Euler-Lagrange conditions can be rewritten as

$$c''(s) = 0$$

The evolution equation

$$\begin{aligned} \frac{dc}{dt} &= \frac{d^2c}{ds^2} \\ &= \kappa N \end{aligned}$$

is also known as the geometric heat equation.
$\kappa$ is the curvature of the curve, $||c''(s)||$.

# Thin-plate spline energy of c(s)

$$E_{TPS}(c) = \int_0^L ||c''(s)||^2 \, ds = \int_0^L x''(s)^2 + y''(s)^2 \, ds$$

For an arclength parameterization, minimizing $E_{TPS}(c)$ is equivalent to minimizing the square curvature of c.

The conditions for minimizing

$$\min_c \int_\Omega F(s, c(s), c'(s), c''(s)) \, ds$$

are

$$F_c - \frac{d}{ds} F_{c'(s)} + \frac{d^2}{ds^2} F_{c''(s)} = 0$$

The conditions for minimizing $E_{TPS}(c)$ are

$$c''''(s) = 0$$

# Constant Coefficients $\alpha, \beta$

Minimizing snake energy

$$\min_{x(s),y(s)} \int_0^1 \frac{1}{2}\alpha(x'(s)^2 + y'(s)^2) + \frac{1}{2}\beta(x''(s)^2 + y''(s)^2) + E_{ext}(x(s), y(s))\; ds$$

The conditions for minimization are

$$\begin{aligned} -\alpha x_{ss} + \beta x_{ssss} + \frac{\partial E_{ext}}{\partial x} &= 0 \\ -\alpha y_{ss} + \beta y_{ssss} + \frac{\partial E_{ext}}{\partial y} &= 0 \end{aligned}$$

The evolution equation is

$$\frac{\partial v}{\partial t} = \alpha v_{ss} - \beta v_{ssss} - \nabla E_{ext}$$

# Non-Constant Coefficients $\alpha(s), \beta(s)$

Minimize $E_{int} + E_{ext}$ with respect to $x(s)$ and $y(s)$:

$$\begin{aligned} \min_{x(s),y(s)} \int_0^1 \quad & \frac{1}{2}\alpha(s)(x'(s)^2 + y'(s)^2) \\ + \quad & \frac{1}{2}\beta(s)(x''(s)^2 + y''(s)^2) \\ + \quad & E_{ext}(x(s), y(s)) \; ds \end{aligned}$$

The conditions are

$$\begin{aligned} -\frac{\partial}{\partial s}(\alpha(s)x'(s)) + \frac{\partial^2}{\partial s^2}(\beta(s)x''(s)) + \frac{\partial E_{ext}}{\partial x} &= 0 \\ -\frac{\partial}{\partial s}(\alpha(s)y'(s)) + \frac{\partial^2}{\partial s^2}(\beta(s)y''(s)) + \frac{\partial E_{ext}}{\partial y} &= 0 \end{aligned}$$

# Outline


1. Basic active contour behavior
   - Geometry of curves
2. Internal forces
   - Membrane spline energy
   - Thin-plate spline energy
3. Image-based forces
4. Constraints
5. Numerical implementation
6. Weaknesses
7. Extensions to the original paper

- The **image** is a mapping $I : R^2 \to R$.
- The **curve** is a mapping $v : R \to R^2$.
- We may form composite functions from these two mappings

**For example:**

- $I(v(p))$ is the intensity of the image at the point $v(p)$ on the curve.
- $\nabla I(v(p))$ is the image gradient at the point $v(p)$ on the curve.

Consider the weighted sum of two energy terms:

$$E_{image} = w_{line} E_{line} + w_{edge} E_{edge}$$

We will ignore the termination functional from the paper.

# Line Functional

$$w_{line}E_{line} = w_{line}I(v(s))$$

- if $w_{line} > 0$ the snake will be attracted to dark contours
- if $w_{line} < 0$ the snake will be attracted to light contours

# Edge Functional

$$E_{edge(1)} = -||\nabla I(v(s))||^2$$

The snake will be attracted to large image gradients.

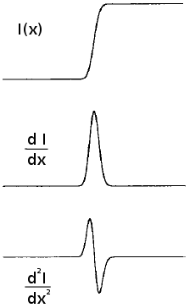
# Scale Space

- We can use scale space to enlarge the convergence region of the snake.
- Recall that linear scale spaces result in blurred edges at coarse scales.
- This will propagate edge information far from the edge.

$$E_{edge(2)} = -||\nabla(G_\sigma * I(v(s)))||^2$$

# Marr-Hildreth

- Edges : zero crossings of the Laplacian ($\nabla^2 = I_{xx} + I_{yy}$)




$$E_{edge(3)} = (G_\sigma * \nabla^2 I(v(s)))^2$$

The snake will be attracted to zero crossings of the smoothed Laplacian.

# Outline


1. Basic active contour behavior
   - Geometry of curves
2. Internal forces
   - Membrane spline energy
   - Thin-plate spline energy
3. Image-based forces
4. Constraints
5. Numerical implementation
6. Weaknesses
7. Extensions to the original paper

# User Applied Constraints

## Spring Energy

The user may connect virtual springs between fixed point $p$ , and contour position $v$:

$$\begin{aligned} E_{spring} &= k||p - v||^2 \\ &= k((p_x - x)^2 + (p_y - y)^2) \end{aligned}$$

where k is a constant (spring stiffness)

$$\nabla E_{spring} = \begin{bmatrix} -2k(p_x - x) \\ -2k(p_y - y) \end{bmatrix} = -2k(p - v)$$

In the evolution equation

$$\frac{\partial v}{\partial t} = \alpha v_{ss} - \beta v_{ssss} - \nabla E_{ext}$$

# User Applied Constraints

## Repulsive Energy

Forces $v$ away from fixed position $p$:

$$
\begin{aligned}
E_{repulsion} &= \frac{1}{||p - v||} \\
&= \frac{1}{\sqrt{(p_x - x)^2 + (p_y - y)^2}}
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial E_{repulsion}}{\partial x} &= (p_x - x)((p_x - x)^2 + (p_y - y)^2)^{-\frac{3}{2}} \\
\frac{\partial E_{repulsion}}{\partial y} &= (p_y - y)((p_x - x)^2 + (p_y - y)^2)^{-\frac{3}{2}}
\end{aligned}
$$

# User Applied Constraints

$$\frac{\partial E_{repulsion}}{\partial x} = (p_x - x)((p_x - x)^2 + (p_y - y)^2)^{-\frac{3}{2}}$$
$$\frac{\partial E_{repulsion}}{\partial y} = (p_y - y)((p_x - x)^2 + (p_y - y)^2)^{-\frac{3}{2}}$$

$$\nabla E_{repulsion} = \frac{1}{r^2}\frac{p - v}{r}$$

where $r = ||p - v||$.
In the evolution equation, this term pushes $v$ in the direction $v - p$ with magnitude $\frac{1}{r^2}$.

# Outline


1. Basic active contour behavior
   - Geometry of curves
2. Internal forces
   - Membrane spline energy
   - Thin-plate spline energy
3. Image-based forces
4. Constraints
5. Numerical implementation
6. Weaknesses
7. Extensions to the original paper

# Discretized Curve

Store the curve as a vector of samples of $v(s)$ at evenly spaced intervals in $s$. For v(s) = $(x(s), y(s))$,

$$\begin{aligned} \mathbf{x}_i &= x(ih) \\ \mathbf{y}_i &= y(ih) \end{aligned}$$

where $h$ is the parameter step size.

Compute derivatives of $v(s)$ using finite difference formulas.

# For non-constant $\alpha(s)$, $\beta(s)$

$$
\begin{aligned}
-\frac{\partial}{\partial s}(\alpha(s)x'(s)) + \frac{\partial^2}{\partial s^2}(\beta(s)x''(s)) + \frac{\partial E_{ext}}{\partial x} &= 0 \\
-\frac{\partial}{\partial s}(\alpha(s)y'(s)) + \frac{\partial^2}{\partial s^2}(\beta(s)y''(s)) + \frac{\partial E_{ext}}{\partial y} &= 0
\end{aligned}
$$

Witkin, Kass, Terzopoulos discretize the Euler-Lagrange equations at this point.
First, they discretize $(x', x'', y', y'')$ using backward and central differences:

$$
\begin{aligned}
-\frac{\partial}{\partial s}(\alpha(s)(x_i - x_{i-1})) + \frac{\partial^2}{\partial s^2}(\beta(s)(x_{i-1} - 2x_i + x_{i+1})) + f_x(i) &= 0 \\
-\frac{\partial}{\partial s}(\alpha(s)(y_i - y_{i-1})) + \frac{\partial^2}{\partial s^2}(\beta(s)(y_{i-1} - 2y_i + y_{i+1})) + f_y(i) &= 0
\end{aligned}
$$

Then, they discretize $(\frac{\partial}{\partial s}, \frac{\partial^2}{\partial s^2})$ using forward and central differences:

$$
\begin{aligned}
- & \quad (\alpha_{i+1}(x_{i+1} - x_i) - \alpha_i(x_i - x_{i-1})) \\
+ & \quad \beta_{i-1}(x_{i-2} - 2x_{i-1} + x_i) \\
- & \quad 2\beta_i(x_{i-1} - 2x_i + x_{i+1}) \\
+ & \quad \beta_{i+1}(x_i - 2x_{i+1} + x_{i+2}) \\
+ & \quad f_x(i) = 0
\end{aligned}
$$

By doing the same for $y$, we can write two linear systems for the snake model...

The two Euler-Lagrange equations can be written in matrix form

$$\begin{aligned} Ax + f_x(x,y) &= 0 \\ Ay + f_y(x,y) &= 0 \end{aligned}$$

where $A$ is $(n \times n)$ sparse matrix with 5 nonzero diagonals.

- $A$ represents the smoothness of the curve
- $f_x, f_y$ represent the external forces

The system of evolution equations is

$$\begin{aligned} \frac{\partial x}{\partial t} &= -Ax - f_x(x,y) \\ \frac{\partial y}{\partial t} &= -Ay - f_y(x,y) \end{aligned}$$

# The linearized evolution equation

The authors present a mixed explicit/implicit method:
**implicit** with respect to internal forces, and
**explicit** with respect to external forces.
Writing the finite difference in time as $\gamma(v^t - v^{t-1})$

$$\begin{aligned} Ax^t + f_x(x^{t-1}, y^{t-1}) &= -\gamma(x^t - x^{t-1}) \\ Ay^t + f_y(x^{t-1}, y^{t-1}) &= -\gamma(y^t - y^{t-1}) \end{aligned}$$

These equations can be rewritten as

$$\begin{aligned} (A + \gamma I)x^t &= \gamma x^{t-1} + f_x(x^{t-1}, y^{t-1}) \\ (A + \gamma I)y^t &= \gamma y^{t-1} + f_y(x^{t-1}, y^{t-1}) \end{aligned}$$

$(A + \gamma I)$ is constant over time, so this matrix may be factorized/inverted once.

# Outline


1. Basic active contour behavior
   - Geometry of curves
2. Internal forces
   - Membrane spline energy
   - Thin-plate spline energy
3. Image-based forces
4. Constraints
5. Numerical implementation
6. Weaknesses
7. Extensions to the original paper

# Weaknesses

- Snakes are prone to getting stuck in local minima (they only see local image data)
- Topologically limited
- Only a 2D model
- Parameterization dependent
- Snakes may self-intersect, or become degenerate.

# Outline


1. Basic active contour behavior
   - Geometry of curves
2. Internal forces
   - Membrane spline energy
   - Thin-plate spline energy
3. Image-based forces
4. Constraints
5. Numerical implementation
6. Weaknesses
7. Extensions to the original paper

# Inflation Force

Also called the "Balloon model"

L. D. Cohen, "On active contour models and balloons", CVGIP: Image Understanding, 1991.

$$\nabla E = f(v(s))N(v(s))$$

- The snake expands in the normal direction.
- $f$ may be intensity-based, edge-based, or constant
- This force can push the snake past local minima of the energy functional

# Adding mass to the snake

$$\mu \frac{\partial^2 v}{\partial t^2} + \gamma \frac{\partial v}{\partial t} = \alpha v_{ss} - \beta v_{ssss} - \nabla E_{ext}$$

- Introducing mass gives the model inertia.
- This model can overshoot local minima.
- Equilibrium when $\frac{\partial^2 v}{\partial t^2} = \frac{\partial v}{\partial t} = 0$

# Reparameterization

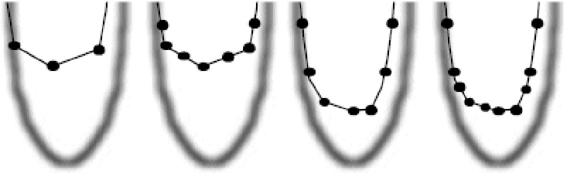
$$E_{param} = \int_{\Omega} (||v'(s)||^2 - c)^2 \; ds$$

- $v'(s) \cdot v'(s) = 1$ for an arclength parameterization.
- This energy can maintain an arclength parameterization
- Some degeneracies can be avoided

Avoided in the level-set formulation by representing the curve/surface implicitly.

# Subdivision

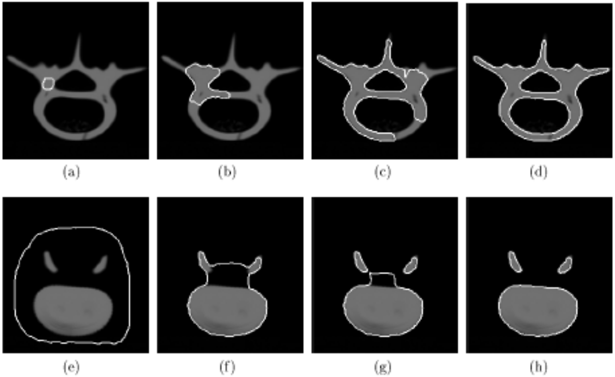
Ivins, J., Porrill, J., "Statistical snakes: Active region models.", Proc. 5th British Machine Vision Conf., 1994.
also proposed by others.



- Add more sample points as $v(s)$ grows longer.
- Reparameterize so that high curvature regions are samples more densely.

# T-snakes

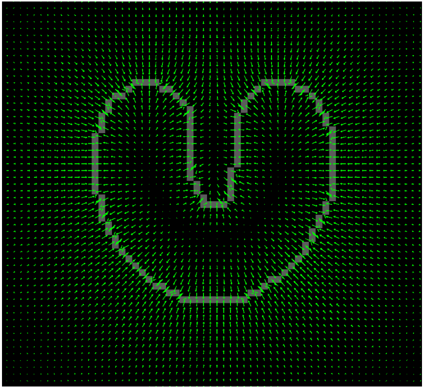
McInerney, T. and Terzopoulos, D., ”T-snakes: Topology adaptive snakes”, Medical Image Analysis, 2000.




The level-set formulation is also topologically adaptive.

# Gradient Vector Field

- The gradient of $I$ does not provide useful information when the image is smooth.
- We would like to know the direction to the nearest edge.



# Gradient Vector Field

Xu, C. and Prince, J.L., ”Gradient Vector Flow: A New External Force for Snakes”, CVPR, 1997.

The gradient vector field, $\mathbf{g}(x, y) = [u(x, y), v(x, y)]$ minimizes the energy

$$E_{gvf} = \int_{\Omega} \mu(u_x^2 + u_y^2 + v_x^2 + v_y^2) + ||\nabla I||^2 ||\mathbf{g} - \nabla I||^2 \; ds$$

- First term: membrane spline smoothness on $u, v$.
- Second term: $\mathbf{g}$ is near $\nabla I$ when $||\nabla I||$ is large
- When $||\nabla I||$ is small, the smoothness term dominates

# Gradient Vector Field

- Determine the evolution equation for minimizing $E_{gvf}$.
- Given $I(x, y)$ compute the gvf, $\mathbf{g}(x, y)$.
- Use $\mathbf{g}(x, y)$ in the evolution equation for $v(s)$.

$$\frac{\partial v}{\partial t} = \alpha v_{ss} - \beta v_{ssss} + \mathbf{g}(v(s))$$

# Next Class

Level Set Methods : implicit active contours.

## Read

Malladi, R., Sethian, J., Vemuri, B., ”Shape Modeling with Front Propagation : A Level Set Approach.”, IEEE PAMI, 1995.